Oct - Dec © 2018 Transactions; 5(20): 1005-1010

International Journal of Research and Applications

ISSN (online): 2349-0020 ISSN (print): 2394-4544 http://www.ijraonline.com/

Research Article

Undecic Spline Approximation of Seventh order Linear Boundary value Problems

Parcha Kalyani

Corresponding Author:
kk.parcha@yahoo.com

DOI:
http://dx.doi.org/
10.17812/IJRA.5.20(2)2018

Manuscript:
Received: 12th Oct, 2018
Accepted: 7th Nov, 2018
Published: 21st Dec, 2018

Publisher:
Global Science Publishing Group, USA
http://www.globalsciencepg.org/

ABSTRACT

Undecic spline approximations are developed following cubic spline Bickley's procedure and applied on linear boundary value problem of order seven. Numerical solutions are computed at different step lengths and absolute errors are calculated. Approximate and exact solutions are compared. Results are tabulated and pictorially illustrated

Keywords: Spline functions; Seventh order boundary value problems; Eleventh degree spline.

Department of Applied Mathematics, Adama Science and Technology University, Ethiopia

IJRA - Year of 2018 Transactions:
Month: October - December
Volume – 5, Issue – 20, Page No's: 1005–1010
Subject Stream: Mathematics

Paper Communication: Author Direct

Paper Reference Id: IJRA-2018: 5(20)1005-1010

DOI: 10.17812/IJRA.5.20(2)2018

International Journal of Research and Applications
Oct - Dec © 2018 Transactions 5(20): 1005-1010

eISSN : 2349 – 0020
pISSN : 2394 – 4544
www.ijraonline.com

MATHEMATICS RESEARCH ARTICLE

Undecic Spline Approximation of Seventh order Linear Boundary value Problems

Parcha Kalyani
Department of Applied Mathematics,
Adama Science and Technology University, Ethiopia
kk.parcha@yahoo.com

ABSTRACT

Undecic spline approximations are developed following cubic spline Bickley's procedure and applied on linear boundary value problem of order seven. Numerical solutions are computed at different step lengths and absolute errors are calculated. Approximate and exact solutions are compared. Results are tabulated and pictorially illustrated

Keywords: Spline functions; Seventh order boundary value problems; Eleventh degree spline.

1. INTRODUCTION

Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, hydrodynamics, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering, and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified fields of applied sciences [1].

Prior to 1950, the computations involved in numerical method to obtain numerical solution of a differential equation were done manually. Later these computations are carried through the calculators followed by digital computers. Due to rapid advances in computing machines like high speed and accuracy, several researchers have been working for developing the numerical methods to obtain the numerical solution of the ordinary or partial differential equations with specified conditions [2, 3]. Initially, the cubic spline technique has been introduced for solving second order two point boundary value problems. Convergence properties of splines has been given in [4].

A solution of Fifth order boundary value problems has been obtained by sixth degree spline method for a set of linear equations whose coefficients form an upper Hessen berg matrix [2]. The applications of cubic spline have been proved positive in the literature [5, 6].

Seventh order boundary value problems generally arise in modeling induction motors with two rotor circuits [7]. The method, variation of parameters is used for solving sixth– order boundary value problems [8]. Fifth order boundary value problems solved using: reproducing Kernel method [9] and non-polynomial spline method [10]. Octic and nonic spline approximations are developed and analyzed to find the numerical solutions of linear seventh boundary value problems [11, 12]. Numerical solution of seventh order boundary value problems has been obtained by using ninth degree spline functions and compared with eighth degree splines in [13].

In the present work we have constructed undecic (eleventh degree) spline function and applied to solve the linear seventh order differential equation with specified boundary conditions. The numerical solutions are constructed for different step lengths.

2. CONSTRUCTION OF UNDECIC SPLINE FUNCTION

Assume that the interval $[x_0, x_n]$ is divided in to n sub intervals with grid points $x_0, x_1, x_2, x_3, \dots x_n$ initial at x_0 and the function $u(x)$ in the interval $[x_0, x_1]$, is represented by eleventh degree spline in the form $s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 + l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + z_0(x - x_0)^{11}$ for the next interval $[x_1, x_2]$ the spline function $s_1(x)$ we add a term $z_1(x - x_0)^{11}$,

Going on in to the next interval $[x_2, x_3]$ add another term $z_2(x - x_0)^{11}$ and so on until we reach x_n hence the function $s(x)$ represents in the form

$$s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 + l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + \sum_{i=0}^{n-1} z_i (x_n - x_i)^{11}. \quad (1)$$

The seventh derivative of the equation (1) is

$$s^{(7)}(x) = 5040l + 40320t(x - x_0) + 181440v(x - x_0)^2 + 604800w(x - x_0)^3 + 1663200 \sum_{i=0}^{n-1} z_i (x_n - x_i)^4 \quad (2)$$

Method of obtaining the solution of seventh order boundary value problem using eleventh degree spline function

Consider the linear seventh order differential equation

$$y^{(7)}(x) + f(x)y(x) = r(x). \quad (3)$$

with boundary conditions

$$y(x_0) = \beta,\ y(x_n) = \gamma,\ y'(x_0) = \beta',\ y'(x_n) = \gamma',\ y''(x_0) = \beta'',\ y''(x_n) = \gamma'',\ y'''(x_0) = \beta''', \quad (4)$$

from (4), and taking spline approximation in (3) at $x = x_i$ *for* $i = 0,1,2,3,4,5,6,\dots,n$ *we get* $(n+8)$ equations in $(n+11)$ unknown $a, b, c, d, g, j, k, l, t, v, w, z_0, z_1, z_2, z_3, z_4, \dots, z_{n-1}$, after determining these unknowns we substitute them in (1) and thus we get the eleventh degree spline approximation of $y(x)$. Putting $x = x_1, x_2, x_3, \dots x_n$ in the spline function thus determined we get the solution at the grid points; Substituting (1) and (2) in the differential equation (3) at $x = x_m$ where $m = 0,1,2,3,\dots n$ and $s(x) = r(x)$ we get

$$af(x_m) + bf(x_m)(x_m - x_0) + cf(x_m)(x_m - x_0)^2 + df(x_m)(x_m - x_0)^3 + gf(x_m)(x_m - x_0)^4 + jf(x_m)(x_m - x_0)^5 + kf(x_m)(x_m - x_0)^6 + l[f(x_m)(x_m - x_0)^7 + 5040] + t[f(x_m)(x_m - x_0)^8 + 40320(x_m - x_0)] + v[f(x_m)(x_m - x_0)^9 + 181440(x_m - x_0)^2] + w[f(x_m)(x_m - x_0)^{10} + 604800(x_m - x_0)^3] + \sum_{i=0}^{n-1} z_i [f(x_m)(x_m - x_i)^{11} + 1663200(x_m - x_0)^4] = s(x_m).$$

From the boundary conditions (5) we obtain

$$y(x_0) = \beta,$$

$$\beta = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 + l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + \sum_{i=0}^{n-1} z_i (x - x_i)^{11},$$

$$a = \beta \quad (6)$$

$$y(x_n) = \gamma,$$

$$a + b(x_n - x_0) + c(x_n - x_0)^2 + d(x_n - x_0)^3 + g(x_n - x_0)^4 + j(x_n - x_0)^5 + k(x_n - x_0)^6 + l(x_n - x_0)^7 + t(x_n - x_0)^8 + v(x_n - x_0)^9 w(x - x_0)^{10} + \sum_{i=0}^{n-1} z_i (x - x_i)^{11} = \gamma. \quad (7)$$

$$y'(x_0) = \beta',$$

$$\beta' = b + 2c(x - x_0) + 3d(x - x_0)^2 + 4g(x - x_0)^3 + 5j(x - x_0)^4 + 6k(x - x_0)^5 + 7l(x - x_0)^6 + 8t(x - x_0)^7 + 9v(x - x_0)^8 + 10w(x - x_0)^9 + 11 \sum_{i=0}^{n-1} z_i (x - x_i)^{10},$$

$$b = \beta'. \quad (8)$$

$$y'(x_n) = \gamma',$$

$$b + 2c(x_n - x_0) + 3d(x_n - x_0)^2 + 4g(x_n - x_0)^3 + 5j(x_n - x_0)^4 + 6k(x_n - x_0)^5 + 7l(x_n - x_0)^6 + 8t(x_n - x_0)^7 + 9v(x_n - x_0)^8 + 10w(x - x_0)^9 + 11 \sum_{i=0}^{n-1} z_i (x - x_i)^{10} = \gamma'. \quad (9)$$

$$y''(x_0) = \beta'',$$

$$\beta'' = 2c + 6d(x - x_0) + 12g(x - x_0)^2 + 20j(x - x_0)^3 + 30k(x - x_0)^4 + 42l(x - x_0)^5 + 56t(x - x_0)^6 + 72v(x - x_0)^7 + +90w(x - x_0)^8 + 110 \sum_{i=0}^{n-1} z_i (x -)^9,$$

$$2c = \beta''. \quad (10)$$

$$y''(x_n) = \gamma'',$$

$$\gamma'' = 2c + 6d(x - x_0) + 12g(x - x_0)^2 + 20j(x - x_0)^3 + 30k(x - x_0)^4 + 42l(x - x_0)^5 + 56t(x - x_0)^6 + 72v(x - x_0)^7 + 90w(x - x_0)^8 + 110 \sum_{i=0}^{n-1} z_i (x - x_i)^9,$$

$$2c + 6d(x_n - x_0) + 12g(x_n - x_0)^2 + 20j(x_n - x_0)^3 + 30k(x_n - x_0)^4 + 42l(x_n - x_0)^5 + 56t(x_n - x_0)^6 + 72v(x_n - x_0)^7 + 90w(x - x_0)^8 + 110 \sum_{i=0}^{n-1} z_i (x - x_i)^9 = \gamma''. \quad (11)$$

$$y'''(x_0) = \beta''',$$

$$\beta''' = 6d + 24g(x - x_0) + 60j(x - x_0)^2 + 120k(x - x_0)^3 + 210l(x - x_0)^4 + 336t(x - x_0)^5 + 504v(x - x_0)^6 + 720w(x - x_0)^7 + 990 \sum_{i=0}^{n-1} z_i (x - x_i)^8,$$

$$6d = \beta'''. \quad (12)$$

From the boundary conditions and taking spline approximation in the linear seventh order boundary value problem at $x = x_i$ for $i = 0,1,2,3,4,5, \dots, n$ and assuming that $z_{n-1} = z_{n-2} = z_{n-3} = z_{n-4}$, we obtain the spline function thus the solution at the grid points.

3. NUMERICAL ILLUSTRATIONS

In this part we consider two linear boundary value problems, and the numerical solution and absolute errors are given at different step lengths. The approximate solution, exact solution and absolute errors at the grid points are summarized in the tabular form and shown graphically.

Problem1 Consider the linear seventh order boundary value problem with constant coefficients

$$u^{(7)}(x) = -u(x) - e^x(35 + 12x + 2x^2), \quad 0 \le x \le 1 \qquad (13)$$

with boundary conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u^{(2)}(0) = 0, \quad u^{(3)}(0) = -3, u(1) = 0, \quad u'(1) = -e, \quad u^{(2)}(1) = -4e. \qquad (14)$$

The exact solution of the problem is

$$u(x) = x(1-x)e^x$$

Approximating $u(x)$ with the spline

$s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 \quad +l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + \sum_{i=0}^{n-1} z_i (x - x_i)^{11}$.

we get $a = 0$, $b = 1$, $c = 0$, *and* $d = -0.5$, *and* the spline fuction $s(x)$ reduces to the form

$s(x) = (x - x_0) - 0.5(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 \quad +l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + \sum_{i=0}^{n-1} z_i (x - x_i)^{11}$.

Solution with h = 0.2

The spline function is

$$s(x) = (x - x_0) - 0.5(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 \quad +l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + \sum_{i=0}^{4} z_i (x - x_i)^{11} \qquad (15)$$

Following the procedure of the method, we get the following values for the unknown coefficients

$g = -0.333333316$ $\quad j = -0.125000079$ $\quad k = -0.033333196$ $\quad l = -0.00694$ $\quad t = -0.001190954$ $\quad v = -0.000172606$ $\quad w = -0.000022938$ $\quad z_0 = -0.00000236317$ $\quad z_1 - 0.00000111989$ $\quad z_2 = -0.00000111989$ $\quad z_3 = -0.00000111989$ $\quad z_4 = -0.0000011198$

Solution with $h = 0.1$

The grid points are $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$ and spline approximation is

$$s(x) = (x - x_0) - 0.5(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 \quad +l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + \sum_{i=0}^{9} z_i (x - x_0)^{11} \qquad (16)$$

Table 1: Numerical solution $S(x)$, Exact solution $u(x)$ and Absolute error of problem 1 with $h = 0.2$

x	S(x)	u(x)	Absolute error
0.2	0.195424441	0.1954244413	9.84662351E − 012
0.4	0.358037927	0.358037927	4.84844941E − 011
0.6	0.437308512	0.437308512	5.61392599E − 011
0.8	0.356086548	0.356086548	1.716249364E − 011

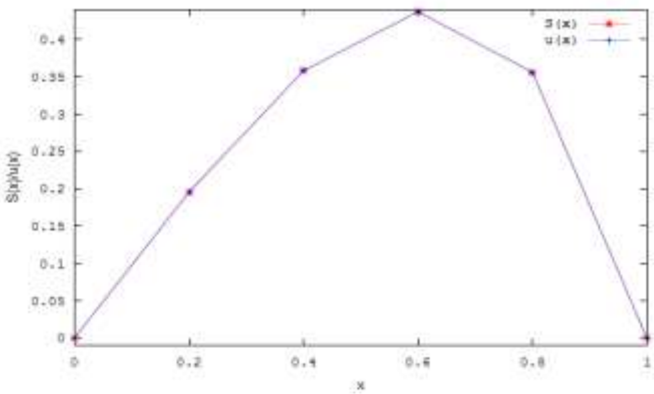

Figure 1: Comparison of approximate solution and exact solution for problem 1 with $h = 0.2$

Solving, we get the values for the unknown coefficients

$g = -0.3333338$ $\quad j = -0.12499156$ $\quad k = -0.03337869$ $\quad l = -0.006944444$ $\quad t = 0.0000242049$ $\quad z_3 = 0.000180440$ $\quad v = -0.006784818$ $\quad w = 0.016170496$ $\quad z_0 = -0.01620906$ $\quad z_1 = 0.017841551$ $\quad z_2 = -0.001800275$ $\quad z_4 = -0.000018549$ $\quad z_5 = 0.00000122277$ $\quad z_6 = -0.000000732981$ $\quad z_7 = -0.000000732981$ $\quad z_8 = -0.000000732981$ $\quad z_9 = -0.000000732981$

Substituting these values in equation (16) we get the numerical solution at the grid points as shown in Table-2.

Numerical solution of problem 1 with $h = 0.2$ and $h = 0.1$ are summarized in table 1 and table 2 respectively. The comparison has been shown in figure 1 and figure 2 .The maximum absolute errors at the given step length are $5.6139259e-011$ and$7.262869e-010$. This is help to understand the agreement between approximate solution and exact solution is good.

Table 2: Numerical solution$S(x)$, exact solution $u(x)$ and Absolute error of the problem 1 with $h = 0.1$

x	***S(x)***	***u(x)***	***Absolute error***
0.1	0.09946538	0.09946538	$4.416939E-12$
0.2	0.19542444	0.19542444	$4.02199E-11$
0.3	0.2834703	0.2834703	$1.79423E-10$
0.4	0.3580379	0.3580379	$3.440639E-10$
0.5	0.4121803	0.4121803	$4.441469E-10$
0.6	0.43730851	0.43730851	$7.262869E-10$
0.7	0.4228880	0.4228880	$3.019809E-10$
0.8	0.3560865	0.3560865	$1.383340E-10$
0.9	0.2213642	0.2213642	$2.413599E-11$

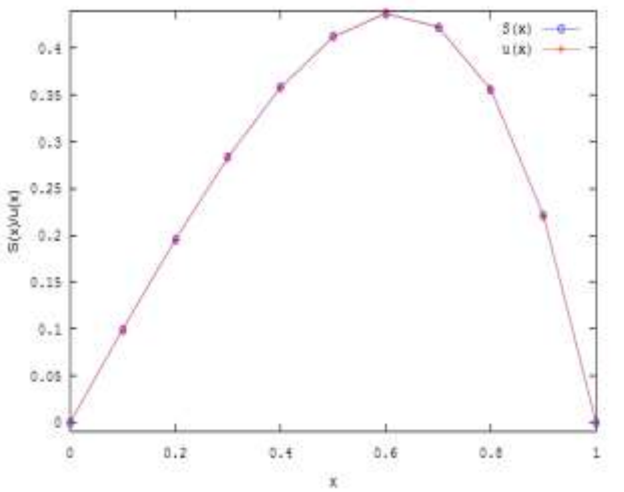

Figure 2: Comparison of approximate solution and exact solution for problem 1 with $h = 0.1$

Problem 2 Seventh order linear boundary value problem with variable coefficients:

$$u^{(7)}(x) = xu(x) + e^x(x^2 - 2x - 6), 0 \le x \le 1 \qquad (17)$$

With the boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \quad u^3(0) = -2,$$
$$u(1) = 0, \quad u'(1) = -e, \quad u''(1) = -2e \qquad (18)$$

The exact solution is $u(x) = (1 - x)e^x$

Solution with $h = 0.2$

The grid points are $x_0, x_1, x_2, x_3, x_4, x_5$ where, $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1$

$$s^7(x_i) - xs(x_i) = e^{x_i}({x_i}^2 - 2x_i - 6),$$

The spline approximation of $u(x)$ using boundary condition is

$s(x) = 1 - 0.5(x - x_0)^2 - 0.333(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6 - 0.0011904\,(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w(x - x_0)^{10} + \sum_{i=0}^{4} z_i\,(x - x_i)^{11}$.

Following the procedure we get the values for unknown coefficients:

$g = -0.1250000007 \quad j = -0.0333333328$
$k = -0.0069444412 \quad l = -0.001190476190$
$t = -0.0001736376 \quad v = -0.0000219820$
$w = -0.000002539 \quad z_0 = -0.0000002478835$
$z_1 = -0.00000009593294$
$z_2 = -0.00000009593294 \quad z_3 = -0.00000009593294 \quad z_4 = -0.000000095932$

Substituting these values in the spline approximation we get the solution at grid points.

Table 3 represnts the approximate solution, exact solution and absolute errors of problem 2 at $h = 0.2$ and comparison has been shown in Figure 3, the maximum absolute error is $1.7867\,e - 01$.

Solution with $h = 0.1$

Table 4 represents the approximate solution, exact solution and absolute errors of problem 2 at $h = 0.1$ and comparison has been shown in Figure 4, the maximum absolute error is$3.157796\,e - 011$.

Table 3: Numerical solutionS(x), exact solution u(x) and Absolute error of problem 2 with h = 0.2

x	*S(x)*	*u(x)*	*Absolute error*
0.2	0.977122206527235	0.97712220652	$9E-13$
0.4	0.895094818576629	0.89509481858	$8E-12$
0.6	0.728847520138337	0.72884752015	$2E-11$
0.8	0.445108185684584	0.44510818569	$1E-11$

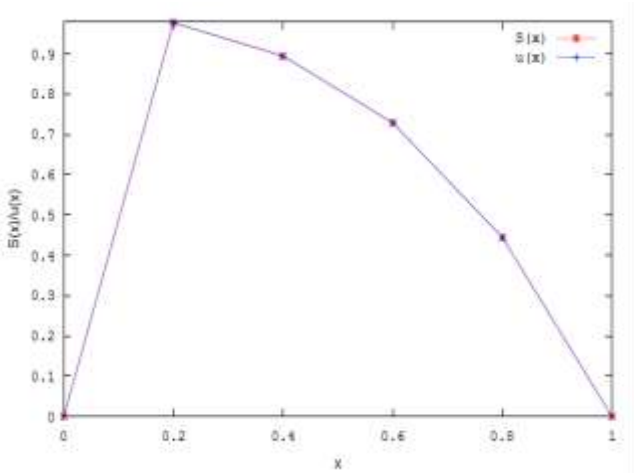

Figure 3: Comparison of approximate and exact solution for problem 2 with $h = 0.2$

Table 4: Numerical solution s(x), exact solution u(x) and absolute errors of problem 2 with $h = 0.1$

x	S(x)	u(x)	Absolute error
0.1	0.994653826	0.994653826	$4.1E-13$
0.2	0.97712220	0.977122206	$2.6E-12$
0.3	0.944901165	0.944901165	$1.3E-11$
0.4	0.895094818	0.895094818	$2.4E-11$
0.5	0.824360635	0.824360635	$3.2E-11$
0.6	0.728847520	0.728847520	$3.1E-11$
0.7	0.604125812	0.604125812	$2.2E-11$
0.8	0.445108185	0.445108185	$1.1E-11$
0.9	0.245960311	0.245960311	$6.6E-14$

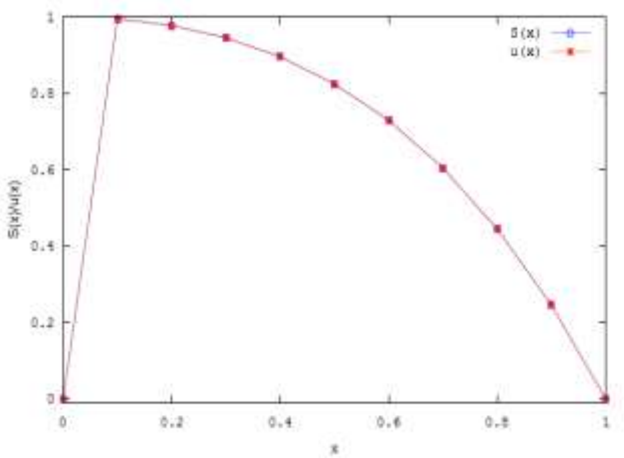

Figure **4**: Comparison of approximate solution and exact solution for problem 2 with $h = 0.1$

4. CONCLUSION

We developed the numerical method to obtain the solution of seventh order boundary value problems using eleventh degree spline function and applied to find the approximate solutions of seventh order boundary value problems by taking two examples. Computational work has been carried out using mathematical software. From the problems 1 and 2 with different step lengths we observed that approximate solutions are more close to the exact solution. It is also observed that the proposed method is well suited for the solution of higher order boundary value problems and reduce the computational work. Therefore the presented method is more accurate and reliable technique for higher boundary value problems. The final conclusion states that when the degree of spline increases the solution is more accurate and also when the step length decreases the solution became more accurate.

REFERENCES

1) J.Rashidina, M.Khazaei, H.Nikmarvani (2015), Spline collocation method for solution of higher order linear boundary value problems TWMS. Pure Appl. Math., V.6, N.1, pp 38-47.

2) Parcha Kalyani and Patibanda S. Rama Chandra Rao, A conventional approach for the solution of the fifth order boundary value problems using sixth degree spline functions, Applied Mathematics, 2013, 4,583-585.

3) Ascher, U. M., Mattheij, R. M. M., and Russell, R. D. Numerical solution of boundary value problems for ordinary differential equations, vol. 13 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1995.

4) H. Ahlberg, E. N. Nilson (1963): Convergence properties of the spline fit. SIAM Journal 11, 95-104.

5) I. J. Schoenberg (1946): Contributions to the problems of approximation of equidistant data by analytic functions. Quart. Appl. Math. 4, 45-99 and 112-141.

6) I. J. Schoenberg (1958): Spline functions, convex curves and mechanical quadrature. Bull. Am. Math. Soc. 64, 352-357.

7) Richards, G. and P.R.R. Sarma, 1994. Reduced order models for induction motors with two rotor Circuits. IEEE Trans. Energ. Conv., 9(4): 673-678.

8) Mohyud-Din, S.T., M.A. Noor and A. Waheed, 2009.Variation of parameters method for solving sixth-order boundary value problems. Comm.Korean Math. Soc., 24(4): 605-615.

9) Akram, G. and H. Rehman, 2011.Solution of fifth order boundary value problems in reproducing kernel space. Middle-East J. Sci. Res., 10(2): 191-195.

10) Siddiqi, S.S. and G. Akram, 2006a, b.Solutions of fifth and sixth order boundary-value problems using non polynomial spline technique. Appl. Math. Comput. 175(2): 1574-1581.

11) Mihretu.N, and P.Kalyani, Eighth Degree spline for seventh order boundary value problems. Journal of Multidisciplinary Engineering science and Technology (JMEST), ISSN: 3159-0040, V0l.2 Issue 5, May-2015.

12) Mihretu.N, P.Kalyani and Dejene B. Comparative study of higher degree spline solutions of linear seventh order boundary value problems International Journal of Mathematical Archive-8(7), 2017, 147 -154.

13) P. Kalyani, Mihretu N. L. Solutions of seventh order boundary value problems using ninth degree spline functions and comparison with eighth degree spline solutions, Journal of applied Mathematics and physics, 2016,4,249-261.